Anti-self-dual Maxwell solutions on hyperkähler manifold and $N = 2$ supersymmetric Ashtekar gravity

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Abstract

Anti-self-dual (ASD) Maxwell solutions on 4-dimensional hyperkähler manifolds are constructed. The $N = 2$ supersymmetric half-flat equations are derived in the context of the Ashtekar formulation of $N = 2$ supergravity. These equations show that the ASD Maxwell solutions have a direct connection with the solutions of the reduced N=2 supersymmetric ASD Yang-Mills equations with a special choice of gauge group. Two examples of the Maxwell solutions are presented.

PACS numbers 04.20.Jb, 04.65.+e

1 Introduction

The Ashtekar formulation of Einstein gravity gives a new insight to the search for anti-self-dual (ASD) solutions without cosmological constant. These are constructed from the solutions of certain differential equations for volume-preserving vector fields on a 4-dimensional manifold. This characterization of the ASD solutions has been originally given by Ashtekar, Jacobson and Smolin [1], and further elaborated by Mason and Newman [2]. In the following, we call their differential equations the half-flat equations. These equations clarify the relationship between the ASD solutions of the Einstein and the Yang-Mills equations. Indeed, if we specialize the gauge group to be a volume-preserving diffeomorphism group, the reduced ASD Yang-Mills equations on the Euclidean space are identical to the half-flat equations [2].

Looked at geometrically, the Ashtekar formulation emphasizes the hyperkähler structures that naturally exist on ASD Einstein solutions. A hyperkähler manifold is a $4n$-dimensional Riemannian manifold (M, g) such that (1) M admits three complex structures $J^a (a = 1, 2, 3)$ which obey the quarternionic relations $J^a J^b = -\delta_{ab} - \epsilon_{abc} J^c$; (2) the metric

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g is preserved by J^a;(3) the 2-forms B^a defined by $B^a(X,Y) = g(J^aX,Y)$ for all vector fields X, Y are three kähler forms, i.e. $dB^a = 0 \quad (a = 1,2,3)$. The solutions of the half-flat equations ensure the conditions above and hence 4-dimensional hyperkähler metrics are ASD Einstein solutions.

Recently, making use of the half-flat equations we have explicitly constructed several hyperkähler metrics [3]. Subsequently here we extend the half-flat equations to the case of $N = 2$ supergravity [1] Our formulation has the advantage that the setting of $N = 2$ supersymmetric Yang-Mills theory is automatically provided. In particular ASD Maxwell solutions on hyperkhäler manifolds are elucidated through the relationship to the reduced $N = 2$ ASD Yang-Mills equations. In the literature [4, 5] the $N = 2$ ASD supergravity has been investigated by using the superfield formulation, but our approach is very different and the results in the present work are more concrete.

In Section 2 we review the half-flat equations. In Section 3 we present a new construction of ASD Maxwell solutions on hyperkähler manifolds and derive the $N = 2$ supersymmetric half-flat equations. Finally, in Section 4 two examples of ASD Maxwell solutions are given.

The following is a summary of the notation used in this paper. The $so(3)$ generators and the Killing form are denoted by $E_a (a = 1,2,3)$ and $\langle\, ,\, \rangle$, respectively. The symbols $\eta^a_{\mu\nu}$ and $\bar{\eta}^a_{\mu\nu}$ $(a = 1,2,3; \mu,\nu = 0,1,2,3)$ represent the 't Hooft matrices satisfying the relations:

$$\eta^a_{\mu\nu} = -\frac{1}{2}\epsilon_{\mu\nu\lambda\sigma}\eta^a_{\lambda\sigma}, \quad \bar{\eta}^a_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\lambda\sigma}\bar{\eta}^a_{\lambda\sigma} \tag{1.1}$$

and

$$\eta^a_{\mu\nu}\eta^b_{\mu\sigma} = \delta_{ab}\delta_{\nu\sigma} + \epsilon_{abc}\eta^c_{\nu\sigma} \quad (\bar{\eta}^a_{\mu\nu} \text{ satisfy the same relations.}). \tag{1.2}$$

In Section 3 we consider a space-time with the signature $(++--)$. Then the metrics $\hat{g}_{\mu\nu} = \mathrm{diag}(1,1,-1,-1)$ and $\kappa_{ab} = \mathrm{diag}(1,-1,-1)$ are used to lower and rise the indices of $\eta^a_{\mu\nu}$ $(\bar{\eta}^a_{\mu\nu})$.

2 Half-flat equations

In this section, we briefly describe the 4-dimendional hyperkähler geometry from the point of view of Ashtekar gravity [1, 2]. We use the metric of the Euclidean signature for avoiding complex variables. The Ashtekar gravity consists of an $so(3)$ connection 1-form $A = A^a \otimes E_a$ and an $so(3)$-valued 2-form $B = B^a \otimes E_a$ on a 4-dimensional manifold M. The action is given by [6]

$$S_{Ash} = \int_M \langle B \wedge F\rangle - \frac{1}{2}\langle C(B) \wedge B\rangle, \tag{2.1}$$

where $F = dA + \frac{1}{2}[A \wedge A]$, $C(B) = C^a{}_b B^b \otimes E_a$ and $C = C^a{}_b E_a \otimes E^b$ is a Lagrange multiplier field which obeys the conditions, $C^a{}_b = C^b{}_a$ and $C^a{}_a = 0$. The equations of

[1] $N = 1$ half-flat equations are obtained from our equations (3.20) $\sim$(3.23) by putting $T = 0$.

motion are

$$F - C(B) = 0 \tag{2.2}$$

$$DB = 0 \tag{2.3}$$

$$B^1 \wedge B^2 = B^2 \wedge B^3 = B^3 \wedge B^1 = 0 \tag{2.4}$$

$$B^1 \wedge B^1 = B^2 \wedge B^2 = B^3 \wedge B^3, \tag{2.5}$$

where D is the covariant derivative with respect to A. The algebraic equations (2.4) and (2.5) represent the constraints of this system.

To solve the constraints we introduce linearly independent four vector fields V_μ ($\mu = 0, 1, 2, 3$) and a volume form ω on M. Then the solutions become the self-dual 2-forms

$$B^a = \frac{1}{2}\bar{\eta}^a_{\mu\nu}\iota_{V_\mu}\iota_{V_\nu}\omega, \tag{2.6}$$

where ι_{V_μ} denotes the inner derivation with respect to V_μ. We proceed to solve the remaining equations (2.2) and (2.3). For the hyperkähler geometry, which we will forcus on in this paper, C must be taken to be zero because ${C^a}_b$ are the coefficients of self-dual Weyl curvature; this is equivalent to the requirement that the holonomy group is contained in subgroup Sp(1) of SO(4). With this choice, (2.2) becomes $F = 0$ and if we take the gauge fixing $A = 0$, (2.3) reduces to

$$dB^a = 0 \quad (a = 1, 2, 3). \tag{2.7}$$

Thus (2.6) implies the half-flat equations [1, 2],

$$\tfrac{1}{2}\bar{\eta}^a_{\mu\nu}[V_\mu, V_\nu] = 0, \tag{2.8}$$

$$L_{V_\mu}\omega = 0. \tag{2.9}$$

This can be seen by applying the formula :

$$d(\iota_X\iota_Y\alpha) = \iota_{[X,Y]}\alpha + \iota_Y L_X\alpha - \iota_X L_Y\alpha + \iota_X\iota_Y d\alpha \tag{2.10}$$

for vector fields X, Y and a form α. Given a solution of (2.8) and (2.9), we have a metric

$$g(V_\mu, V_\nu) = \phi\delta_{\mu\nu}, \tag{2.11}$$

where $\phi = \omega(V_0, V_1, V_2, V_3)$. This metric is invariant by the three complex structures

$$J^a(V_\mu) = \bar{\eta}^a_{\nu\mu}V_\nu \quad (a = 1, 2, 3), \tag{2.12}$$

which obey the relations $J^aJ^b = -\delta_{ab} - \epsilon_{abc}J^c$ and give the three Kähler forms $B^a(V_\mu, V_\nu) = g(J^a(V_\mu), V_\nu)$. Thus the triplet (M, g, J^a) is a hyperkähler manifold. Conversely, it is known that every 4-dimensional hyperkähler manifold locally arises by this construction [2, 7].

This formulaton yields that the vector fields V_μ may be identified with the components of a space-time independent ASD Yang-Mills connection on $\mathbb{R}^4$. Indeed, (2.9) is the

assertion that the gauge group is the diffeomorphism group $\mathrm{SDiff}_\omega(M)$ preserving the volume form ω, and (2.8) are explicitly written as

$$[V_0, V_1] + [V_2, V_3] = 0 \tag{2.13}$$
$$[V_0, V_2] + [V_3, V_1] = 0 \tag{2.14}$$
$$[V_0, V_3] + [V_1, V_2] = 0, \tag{2.15}$$

which are equivalent to the reduced ASD Yang-Mills equations [2].

3 N=2 supersymmetric Ashtekar gravity

We start with the chiral action for $N = 2$ supergravity without cosmological constant [9, 10]. The bosonic part, which is the chiral action of Einstein-Maxwell theory, contains a $U(1)$ connection 1-form a and a 2-form b in addition to A, B in (2.1) [8]. The fermionic fields (two gravitino fields) are expressed by Weyl spinor 1-forms ψ^i and Weyl spinor 2-forms χ^i, where $i(= 1, 2)$ is a Sp(1) index representing the two supersymmetric charges. By using the 2-component spinor notation, the chiral action is written as [2].

$$S^{N=2}_{Ash} = \int B^{AB} \wedge F_{AB} + b \wedge f + \chi^i{}_A \wedge D\psi_i{}^A - \frac{1}{2} b \wedge b - \frac{1}{8} b \wedge \psi_i{}^A \wedge \psi^i{}_A$$
$$-\frac{1}{2} C_{ABCD} B^{AB} \wedge B^{CD} - \kappa^i{}_{ABC} B^{AB} \wedge \chi_i{}^C - \frac{1}{2} H_{AB}(B^{AB} \wedge b - \chi_i{}^A \wedge \chi^{iB}) \tag{3.1}$$

where $f = da$, and $C_{ABCD}, \kappa^i{}_{ABC}$ and H_{AB} are totally symmetric Lagrange multiplier fields.

Let us forcus on ASD solutions. Then we can put $A_{AB} = C_{ABCD} = 0$ as stated in Sect.2, and further impose the conditions $H_{AB} = \kappa^i{}_{ABC} = \psi_i{}^A = 0$. It should be noticed that these restrictions preserve the $N = 2$ supersymmetry; as we will see in Sect.3.2 this symmetry is properly realized in the $N = 2$ supersymmetric ASD Yang-Mills equations with the gauge group $\mathrm{SDiff}_\omega(M)$. Now the equations of motion derived from $S^{N=2}_{Ash}$ reduce to

$$f = b \tag{3.2}$$
$$dB^{AB} = db = d\chi_i{}^A = 0 \tag{3.3}$$
$$B^{(AB} \wedge B^{CD)} = 0 \tag{3.4}$$
$$B^{(AB} \wedge \chi_i{}^{C)} = 0 \tag{3.5}$$
$$B^{AB} \wedge b - \chi_i{}^A \wedge \chi^{iB} = 0. \tag{3.6}$$

3.1 Maxwell solutions on hyperkähler manifolds

We first consider the bosonic sector $(b = f, B)$ in a space-time with the Euclidean signature. The relevant equations are obtained from (3.2)$\sim$(3.6) by putting $\chi_i{}^A = 0$. In the

[2]We have re-named the variables in [10] as $(A_{AB}, A, \psi_\alpha{}^A, \Sigma^{AB}, B, \chi_\alpha{}^A, \Psi_{ABCD}, \kappa^\alpha{}_{ABC}, \phi_{AB}) \mapsto (A_{AB}, a, \frac{1}{\sqrt{2}}\psi_i{}^A, -B^{AB}, -\frac{1}{2}b, -\sqrt{2}\chi_i{}^A, -C_{ABCD}, -\frac{1}{\sqrt{2}}\kappa^i{}_{ABC}, -\frac{1}{2}H_{AB})$

previous section we have seen that the solutions $B^a (a = 1, 2, 3)$ are self-dual Kähler forms on a hyperkähler manifold M. Thus the equations (3.3) and (3.6) imply that b is an ASD closed 2-form (ASD Maxwell solution) on M. The following proposition holds.

Proposition. Let M be a hyperkähler manifold expressed by linear independent vector fields $V_\mu (\mu = 0, 1, 2, 3)$ and a volume form ω as mensioned in (2.8) and (2.9). If the vector field $T = T_\mu V_\mu$ satisfies

$$L_T \omega = 0, \tag{3.7}$$

$$[V_\mu, [V_\mu, T]] = 0, \tag{3.8}$$

then b defined by

$$b = \frac{1}{2} b^a \eta^a_{\mu\nu} \iota_{V_\mu} \iota_{V_\nu} \omega \quad \text{for} \quad b^a = \eta^a_{\mu\nu} V_\mu T_\nu, \tag{3.9}$$

is an ASD closed 2-form on M.

Proof. The ASD condition of b immediately follows from (3.9). Therefore it suffices to prove that b is a closed 2-form. Using the identity of the 't Hooft matrices

$$\eta^a_{\mu\nu} \eta^a_{\lambda\sigma} = \delta_{\mu\lambda}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\lambda} - \epsilon_{\mu\nu\lambda\sigma}, \tag{3.10}$$

we rewrite (3.9) in the form,

$$b = \iota_{V_\mu} \iota_{V_\nu} L_{V_\mu}(T_\nu \omega) - \frac{1}{2} \epsilon_{\mu\nu\lambda\sigma} \iota_{V_\mu} \iota_{V_\nu} L_{V_\lambda}(T_\sigma \omega). \tag{3.11}$$

Let us define the vector fields

$$W_{\mu\nu} = [V_\mu, V_\nu] + \frac{1}{2} \epsilon_{\mu\nu\lambda\sigma} [V_\lambda, V_\sigma]. \tag{3.12}$$

Then,

$$b + \iota_{V_\mu} \iota_{W_{\mu\nu}} T_\nu \omega = \iota_{V_\mu} L_{V_\mu} \iota_T \omega + \frac{1}{2} \epsilon_{\mu\nu\lambda\sigma} \iota_{V_\mu} L_{V_\lambda} \iota_{V_\sigma}(T_\nu \omega). \tag{3.13}$$

The exterior derivative of (3.13) is evaluated as follows: Since both the vector fields V_μ and T preserve the volume form ω, we have

$$\begin{aligned} d(\iota_{V_\mu} L_{V_\mu} \iota_T \omega) &= L_{V_\mu} L_{V_\mu} \iota_T \omega \\ &= \iota_{[V_\mu, [V_\mu, T]]} \omega \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} d(\epsilon_{\mu\nu\lambda\sigma} \iota_{V_\mu} L_{V_\lambda} \iota_{V_\sigma}(T_\nu \omega)) &= \frac{1}{2} \epsilon_{\mu\nu\lambda\sigma} (L_{V_\mu} L_{V_\lambda} \iota_{V_\sigma} - \iota_{V_\mu} L_{V_\lambda} L_{V_\sigma})(T_\nu \omega) \\ &= \frac{1}{4} \epsilon_{\mu\nu\lambda\sigma} \iota_{[[V_\mu, V_\lambda], V_\sigma]} T_\nu \omega = 0. \end{aligned} \tag{3.15}$$

We thus find

$$d(b + \iota_{V_\mu} \iota_{W_{\mu\nu}} T_\nu \omega) = \iota_{[V_\mu, [V_\mu, T]]} \omega. \tag{3.16}$$

Finally, making use of (2.8), i.e. $W_{\mu\nu} = 0$, combined with the condition (3.8), we obtain the required formula $db = 0$. □

Remark. Using the hyperkähler metric (2.11), we can rewrite (3.9) as

$$b = dg(T,\ ) + \iota_{[V_\mu,[V_\mu,T]]}\omega. \tag{3.17}$$

This expression is convenient to the explicit calculation in Sect.4.

3.2 $N = 2$ supersymmetric half-flat equations

Let us return to the equations (3.2)∼(3.6) and assume a space-time with the signature $(++--)$. It is known that the hyperkähler manifolds with this signature provide the consistent backgrounds for closed N=2 strings [11, 12]. We follow the paper for the spinor notation of [4]; the spinor indices $A, B, C\cdots$ in (3.1) are replaced by the dotted indices $\dot{A}, \dot{B}, \dot{C}\cdots$. To solve the constraints we introduce spinor valued vector fields V_i^A in addition to the vector fields V_μ (or $V_{A\dot{B}}$) and T. Referring to (3.9), we put

$$\chi_{i\dot{A}} = \iota_{V_{B\dot{A}}}\iota_{V_i{}^B}\omega, \tag{3.18}$$

$$b = \tfrac{1}{2}(\eta^{a\lambda\sigma}V_\lambda T_\sigma)\eta_a{}^{\mu\nu}\iota_{V_\mu}\iota_{V_\nu}\omega + \iota_{V^i{}_A}\iota_{V_i{}^A}\omega, \tag{3.19}$$

together with (2.6), i.e. $B_{\dot{A}\dot{B}} = \frac{1}{2}\iota_{V_{C\dot{A}}}\iota_{V^C{}_{\dot{B}}}\omega$ in the spinor notation (See Sect.1 for the 't Hooft matrices.). It is easily confirmed that these formulas automatically satisfy (3.4)∼(3.6). Furthermore (3.3) requires the following equations for the vector fields, which are proved in a similar fashion to the preceding proposition:

$$\tfrac{1}{2}\bar{\eta}^{a\mu\nu}[V_\mu, V_\nu] = 0 \tag{3.20}$$

$$[V^\mu, [V_\mu, T]] + [V^i{}_A, V_i{}^A] = 0 \tag{3.21}$$

$$[V_{B\dot{A}}, V_i{}^B] = 0 \tag{3.22}$$

and

$$L_{V_\mu}\omega = L_{V_i{}^A}\omega = L_T\omega = 0. \tag{3.23}$$

This result is satisfactory in that it gives the direct correspondence between the ASD solutions of the $N = 2$ supergravity and the $N = 2$ supersymmetric Yang-Mills theory; the equations (3.20)∼(3.23) can be regarded as $N = 2$ supersymmetric extension of the half-flat equations. To say more precisely, let us recall the $N = 2$ ASD Yang-Mills equation in a flat space-time with the signature $(++--)$ [4, 5]. The $N = 2$ Yang-Mills theory has the field content $(A_\mu, \lambda_{iA}, \widetilde{\lambda}_{i\dot{A}}, S, \widetilde{S})$, where λ_{iA} and $\widetilde{\lambda}_{i\dot{A}}$ are chiral and anti-chiral Majorana-Weyl spinors, while the fields S and $\widetilde{S}$ are real scalors. All the fields are in the adjoint representation of gauge group. By the supersymmetric ASD conditoin, i.e. $\widetilde{S} = 0$, the equations of motion reduce to

$$\tfrac{1}{2}\bar{\eta}^{a\mu\nu}[D_\mu, D_\nu] = 0 \tag{3.24}$$

$$D^\mu D_\mu S + [\lambda^i{}_A, \lambda_i{}^A] = 0 \tag{3.25}$$

$$(\sigma^\mu D_\mu)_{B\dot{A}}\lambda_i{}^B = 0, \tag{3.26}$$

where $D_\mu = \partial_\mu + [A_\mu,\ ]$. If we require thet the fields are all constant on the space-time, and further choose the gauge group as $\mathrm{SDiff}_\omega(M)$, then the equations (3.24)∼(3.26) just become the $N = 2$ supersymmetric half-flat equations (3.20)∼(3.23) with the identification $A_\mu = V_\mu$, $\lambda_i{}^A = V_i{}^A$ and $S = T$.

4 Examples of ASD Maxwell solutions

As an application of the proposition, we present two examples of ASD Maxwell solutions on 4-dimensional hyperkähler manifolds with one isometry generated by a Killing vector field $K = \frac{\partial}{\partial \tau}$. The first example gives the well-known Maxwell solution and the second one leads to a new solution as far as the authors know. We use local coordinates (τ, x^1, x^2, x^3) and a volume form $\omega = d\tau \wedge dx^1 \wedge dx^2 \wedge dx^3$ for the background manifold.

4.1 Gibbons-Hawking background

In this case we choose the vector fields V_μ as [13]

$$V_0 = \phi \frac{\partial}{\partial \tau}, \tag{4.1}$$

$$V_i = \frac{\partial}{\partial x^i} + \psi_i \frac{\partial}{\partial \tau} \quad (i = 1, 2, 3), \tag{4.2}$$

where the functions ϕ, ψ_i are all independent of τ. Then these vector fields preserve the volume form ω and (2.8) implies

$$*d\phi = d\psi, \tag{4.3}$$

where $\psi = \psi_i dx^i$ and $*$ denotes the Hodge operator on $\mathbb{R}^3 = \{(x^1, x^2, x^3)\}$. The resultant metric is the Gibbons-Hawking multi-center metric [14],

$$ds^2 = \phi^{-1}(d\tau + \psi)^2 + \phi dx^i dx^i. \tag{4.4}$$

The Killing vector field $T = K$ clearly satisfies (3.7) and (3.8). Applying the proposition, we have an ASD Maxwell solution [15],

$$b = da \quad \text{with} \quad a = \phi^{-1}(d\tau - \psi). \tag{4.5}$$

4.2 Real heaven background

We choose the vector fields V_μ as [3]

$$V_0 = e^{\frac{\psi}{2}} \left(\partial_3 \psi \cos\left(\frac{\tau}{2}\right) \frac{\partial}{\partial \tau} + \sin\left(\frac{\tau}{2}\right) \frac{\partial}{\partial x^3} \right) \tag{4.6}$$

$$V_1 = e^{\frac{\psi}{2}} \left(-\partial_3 \psi \sin\left(\frac{\tau}{2}\right) \frac{\partial}{\partial \tau} + \cos\left(\frac{\tau}{2}\right) \frac{\partial}{\partial x^3} \right) \tag{4.7}$$

$$V_2 = \frac{\partial}{\partial x^1} + \partial_2 \psi \frac{\partial}{\partial \tau} \tag{4.8}$$

$$V_3 = \frac{\partial}{\partial x^2} - \partial_1 \psi \frac{\partial}{\partial \tau}, \tag{4.9}$$

If the function ψ is independent of τ and satisfies the 3-dimensional continual Toda equation:

$$\partial_1{}^2 \psi + \partial_2{}^2 \psi + \partial_3{}^2 e^\psi = 0, \tag{4.10}$$

these vector fields are solutions of the half-flat equations (2.8) and (2.9). Then, the hyperkähler metric (the real heaven solution) is given by [16]

$$ds^2 = (\partial_3\psi)^{-1}(d\tau + \beta)^2 + (\partial_3\psi)\gamma_{ij}dx^i dx^j, \tag{4.11}$$

where

$$\beta = -\partial_2\psi dx^1 + \partial_1\psi dx^2, \tag{4.12}$$

and γ_{ij} is the diagonal metric $\gamma_{11} = \gamma_{22} = e^{\psi}, \gamma_{33} = 1$.

In this case we find asolution of (3.7) and (3.8):

$$T = c_1(\partial_1\psi)\frac{\partial}{\partial\tau} + c_2(\partial_2\psi)\frac{\partial}{\partial\tau} \quad \text{for constants } c_i \ (i = 1, 2). \tag{4.13}$$

The corresponding ASD Maxwell solution is given by

$$b = c_1 da^{(1)} + c_2 da^{(2)}, \tag{4.14}$$

where

$$a^{(1)} = \partial_1\psi(\partial_3\psi)^{-1}(d\tau + \beta) + \partial_3 e^{\psi}dx^2 - \partial_2\psi dx^3, \tag{4.15}$$
$$a^{(2)} = \partial_2\psi(\partial_3\psi)^{-1}(d\tau + \beta) - \partial_3 e^{\psi}dx^1 + \partial_1\psi dx^3. \tag{4.16}$$

Acknowledgments

We want to thank Y. Hashimoto for many useful discussions.

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